**Problem set 6**

Due date: 18th March

(Submit any five)

**Exercise 52.** Let $X_n, X$ be random variables on a common probability space.

(1) If $X_n \xrightarrow{P} X$, show that some subsequence $X_{n_k} \xrightarrow{a.s.} X$.

(2) If every subsequence of $X_n$ has a further subsequence that converges almost surely to $X$, show that $X_n \xrightarrow{P} X$.

**Exercise 53.** For $\mathbb{R}^d$-valued random vectors $X_n, X$, we say that $X_n \xrightarrow{P} X$ if $\mathbf{P}(\|X_n - X\| > \delta) \to 0$ for any $\delta > 0$ (here you may take $\|\cdot\|$ to denote the usual norm, but any norm on $\mathbb{R}^d$ gives the same definition).

(1) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, show that $(X_n, Y_n) \xrightarrow{P} (X, Y)$.

(2) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, show that $X_n + Y_n \xrightarrow{P} X + Y$ and $X_n Y_n \xrightarrow{P} XY$. [**Hint:** Show more generally that $f(X_n, Y_n) \xrightarrow{P} f(X, Y)$ for any continuous function $f$ by using the previous problem for random vectors].

**Exercise 54.** (1) If $X_n, Y_n$ are independent random variables on the same probability space and $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (U, V)$ where $U \stackrel{d}{=} X$, $V \stackrel{d}{=} Y$ and $U, V$ are independent.

(2) If $X_n \xrightarrow{d} X$ and $Y_n - X_n \xrightarrow{P} 0$, then show that $Y_n \xrightarrow{d} X$.

**Exercise 55.** Show that the sequence $X_n$ is tight if and only if $c_n X_n \xrightarrow{P} 0$ whenever $c_n \to 0$.

**Exercise 56.** Suppose $X_n$ are i.i.d with $\mathbf{E}[|X_1|^4] < \infty$. Show that there is some constant $C$ (depending on the distribution of $X_1$) such that $\mathbf{P}\left(|n^{-1}S_n - \mathbf{E}[X_1]| > \delta\right) \le Cn^{-2}$. (What is your guess if we assume $\mathbf{E}[|X_1|^6] < \infty$? You don't need to show this in the homework).

**Exercise 57.** (1) (**Skorokhod's representation theorem**) If $X_n \xrightarrow{d} X$, then show that there is a probability space with random variables $Y_n, Y$ such that $Y_n \stackrel{d}{=} X_n$ and $Y \stackrel{d}{=} X$ and $Y_n \xrightarrow{a.s.} Y$. [**Hint:** Try to construct $Y_n, Y$ on the canonical probability space $([0,1], \mathcal{B}, \mathbf{m})$]

(2) If $X_n \xrightarrow{d} X$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, show that $f(X_n) \xrightarrow{d} f(X)$. [**Hint:** Use the first part]

**Exercise 58.** Suppose $X_i$ are i.i.d with the Cauchy distribution (density $\pi^{-1}(1+x^2)^{-1}$ on $\mathbb{R}$). Note that $X_1$ is not integrable. Then, show that $\frac{S_n}{n}$ does not converge in probability to any constant. [**Hint:** Try to find the probability $\mathbf{P}(X_1 > t)$, and then use it].